

WORMHOLES IN THE BRANEWORLD

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We discuss brane wormhole solution when classical brane action contains 4d curvature. The equations of motion for the cases with $R = 0$ and $R \neq 0$ are obtained. Their numerical solutions corresponding to wormhole are found for specific boundary conditions.

The wormhole is very important object in modern cosmology. It is suggested that wormholes are bridges linking two distinct spacetimes, or handles between remote parts of a single universe, see for example [1],[2],[3]. Wormhole is the solution of the Einstein equation if the stress-energy tensor of matter violates the null energy conditions (NEC) [3]. There are different ways of evading these violations. Most of these attempts focus on alternative gravity theories or existing of exotic matter [6], [5], [4]. We consider the wormholes in frames of brane worlds [7]. The brane world scenario assumes that our Universe is four-dimensional space-time, embedded in the 5D bulk spacetimes. According to this concept the 4d Einstein equations will be modified if we use the Gauss and Codacci equations [8]. The purpose of this note is to construct brane wormholes in 5d space.

Let $g_{\mu\nu}$ be the metric of the bulk space and n_μ is the unit vector normal to the 3-brane. Then metric induced on the brane has the form:

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu. \quad (1)$$

We start from action:

$$S = \int d^5x \sqrt{-g} \left[\frac{1}{k_5^2} R^{(5)} - 2\Lambda \right] + S_{brane}(q). \quad (2)$$

Here suffix (5) denotes the 5d quantities. The Λ is 5d cosmological constant and $S_{brane}(q)$ is the action on the brane. The bulk Einstein equation is given by

$$\frac{1}{k_5^2} \left(R_{\mu\nu}^{(5)} - \frac{1}{2} g_{\mu\nu} R^{(5)} \right) = T_{\mu\nu}. \quad (3)$$

We assume the 5d metric to have the form

$$ds^2 = d\xi^2 + q_{\mu\nu} dx^\mu dx^\nu. \quad (4)$$

Then the 5d stress-energy tensor takes the following form

$$T_{\mu\nu} = -\Lambda g_{\mu\nu} + (-\lambda q_{\mu\nu} + \tau_{\mu\nu}) \delta(\xi). \quad (5)$$

Here λ is 4d cosmological constant and $\tau_{\mu\nu}$ represents the contribution due to brane matter.

In this case the brane Einstein equation can be represented in the form

$$\begin{aligned} \frac{1}{k_5^2} G_{\mu\nu}^{(4)} = & -\frac{1}{2} \left(\Lambda + \frac{k_5^2 \lambda^2}{6} \right) q_{\mu\nu} + \\ & \frac{k_5^2 \lambda}{6} \tau_{\mu\nu} + k_5^2 \pi_{\mu\nu} - \frac{1}{k_5^2} E_{\mu\nu}. \end{aligned} \quad (6)$$

Here $E_{\mu\nu}$ is the part of the 5d Weyl tensor defined by

$$E_{\mu\nu} = C_{\alpha\beta\gamma\delta}^{(5)} n^\alpha n^\beta q_\mu^\gamma q_\nu^\delta,$$

and $\pi_{\mu\nu}$ is given by

$$\pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\alpha} \tau_\nu^\alpha + \frac{1}{12} \tau \tau_{\mu\nu} + \frac{1}{8} q_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{24} q_{\mu\nu} \tau^2.$$

Let us consider the brane action to be the following form

$$S_{brane} = \int d^4x \sqrt{-q} \left(-\alpha R^{(4)}(q) - 2\lambda \right). \quad (7)$$

Here α is arbitrary parameter. In this case the brane equation is taken as following [9]:

$$\begin{aligned} \frac{1}{\kappa_5^2} \left(1 - \frac{k_5^4 \lambda \alpha}{6} \right) \left(R_{\alpha\beta}^{(4)} - \frac{R^{(4)}}{2} q_{\alpha\beta} \right) = \\ -\frac{1}{2} \left(\Lambda + \frac{k_5^2 \lambda^2}{6} \right) q_{\alpha\beta} + \alpha^2 \kappa_5^2 \left[-\frac{1}{4} R_{\alpha\mu}^{(4)} R_\beta^{(4)\mu} + \right. \\ \left. \frac{1}{6} R^{(4)} R_{\alpha\beta}^{(4)} - q_{\alpha\beta} \left(\frac{1}{16} R^{(4)2} - \frac{1}{8} R_{\alpha\beta}^{(4)} R^{(4)\alpha\beta} \right) \right] - \\ \frac{1}{\kappa_5^2} C_{\mu\nu\rho\delta}^{(5)} n^\mu n^\rho q_\alpha^\nu q_\beta^\delta. \end{aligned} \quad (8)$$

Let us consider the static, spherically symmetric metric on the brane:

$$\begin{aligned} ds^2 = & -e^{2a_1(r)} dt^2 + e^{2a_2(r)} dr^2 + \\ & r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \quad (9)$$

The 4d curvature for metric (9) has the form

$$\begin{aligned} R^{(4)} = & \frac{2e^{-2a_2}}{r^2} (-1 + e^{2a_2} - r^2 \dot{a}_1^2 + \\ & 2r\dot{a}_2 + r\dot{a}_1(-2 + r\dot{a}_2) - r^2 \ddot{a}_1). \end{aligned} \quad (10)$$

It is simple to find the 4d combination

$$\begin{aligned} R_{\alpha\beta}^{(4)} R^{(4)\alpha\beta} = \\ e^{-4a_2} \left(\frac{2[-1 + e^{2a_2} - r\dot{a}_1 + r\dot{a}_2]^2}{r^4} + \right. \\ \left. \left[-\dot{a}_1^2 + \frac{2\dot{a}_2}{r} + \dot{a}_1\dot{a}_2 - \ddot{a}_1 \right]^2 + \right. \\ \left. \frac{[r\dot{a}_1^2 + \dot{a}_1(2 - r\dot{a}_2) + r\ddot{a}_1]^2}{r^2} \right) \end{aligned} \quad (11)$$

If we consider the situation when $R^{(4)} = 0$, one obtains the equation:

$$\ddot{a}_1 = \frac{-1 + e^{2a_2} - r^2 \dot{a}_1^2 + 2r\dot{a}_2 + r\dot{a}_1(-2 + r\dot{a}_2)}{r^2}. \quad (12)$$

Now we rewrite the Eq. (11) in the form:

$$R_{\alpha\beta} R^{\alpha\beta} = \frac{e^{-4a_2}}{r^4} \left(\frac{3}{2} [b_1^2 + b_2^2] + b_1 b_2 \right). \quad (13)$$

Here $b_1 = -1 + e^{2a_2} + 2r\dot{a}_2$, $b_2 = -1 + e^{2a_2} - 2r\dot{a}_1$. Since the Weyl tensor is traceless:

$$R_{\alpha\beta}^{(4)} R^{(4)\alpha\beta} = \frac{8}{\alpha^2} \left(\frac{\Lambda}{\kappa_5^2} + \frac{\lambda^2}{6} \right). \quad (14)$$

If Λ is zero and λ is zero, we obtain the Schwarzschild metric, but this metric is Ricci flat:

$$ds^2 = -c_1 \left(\frac{1}{c_2} - \frac{1}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{c_2}{r}} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2).$$

It is easy to see, that if λ is zero, then Λ one should be positive.

One can see, that the Ricci tensor has the form:

$$R_{\beta}^{(4)\alpha} = \frac{e^{-2a_2}}{r^2} \text{diag} \left(-b_1, -b_2, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2} \right).$$

The Eq. (11) can be rewritten in the form:

$$\begin{aligned} \frac{1}{\kappa_5^2} \left(1 - \frac{k_5^4 \lambda \alpha}{6} \right) R_{\alpha}^{(4)\mu} q_{\mu\beta} &= \frac{1}{2} \left(\Lambda + \frac{k_5^2 \lambda^2}{6} \right) q_{\alpha\beta} - \\ \frac{\alpha^2 \kappa_5^2}{4} R_{\alpha}^{(4)\mu} q_{\mu\nu} R_{\beta}^{(4)\nu} &- \frac{1}{\kappa_5^2} E_{\alpha\beta}. \end{aligned}$$

The classical Einstein equation in 4 dimensions is taken as following:

$$R_{\alpha\beta}^{(4)} - \frac{1}{2} R^{(4)} q_{\alpha\beta} = k_4^2 T_{\alpha\beta}, \quad (15)$$

Or

$$R_{\alpha\beta}^{(4)} = k_4^2 T_{\alpha\beta}. \quad (16)$$

Hence the stress-energy tensor looks like:

$$\begin{aligned} T_{\alpha\beta} &= \kappa_5^2 \left(1 - \frac{k_5^4 \lambda \alpha}{6} \right)^{-1} \left(\frac{1}{2} \left(\Lambda + \frac{k_5^2 \lambda^2}{6} \right) q_{\alpha\beta} + \right. \\ &\quad \left. - \frac{\alpha^2 \kappa_5^2}{4} R_{\alpha}^{(4)\mu} q_{\mu\nu} R_{\beta}^{(4)\nu} - \frac{1}{\kappa_5^2} E_{\alpha\beta} \right). \end{aligned}$$

In order to find the solution of the Eqs. (12-14) let us consider the following of b_1 and b_2

$$b_1 = r^2 e^{2a_2} f_1(r), \quad (17)$$

$$b_2 = r^2 e^{2a_2} f_2(r). \quad (18)$$

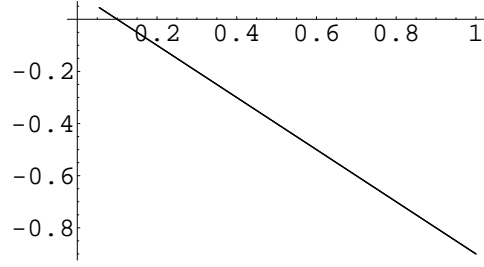


Figure 1: Evolution of the function $f(r)$.

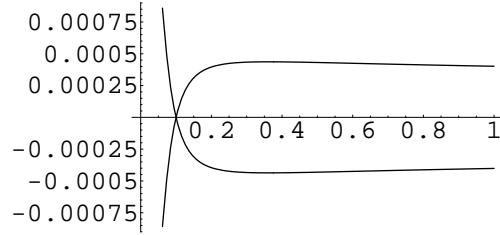


Figure 2: Evolution of the function $f_2(r)$. Here $c = 1$, $\beta = 0.001$.

Then we obtain the equation on the function $f_1(r)$ and $f_2(r)$

$$\frac{3}{2} (f_1(r)^2 + f_2(r)^2) + f_1(r) f_2(r) = \beta^2 \quad (19)$$

$$\begin{aligned} -2c + r + r^5 f_2(r)^2 - 2cr^3 \dot{f}_2(r) - 3c \dot{f}(r) + r \dot{f}(r) + \\ f(r) [3 + 2r^2 \dot{f}_2(r) + 3 \dot{f}(r)] - \\ r^2 f_2(r) [5c + 2r - 5f(r) + r \dot{f}(r)] = 0, \end{aligned} \quad (20)$$

$$f_1(r) = \frac{1 + \dot{f}(r)}{r^2}. \quad (21)$$

Where c is the constant of integration and $\beta^2 = \frac{8}{\alpha^2} \left(\frac{\Lambda}{\kappa_5^2} + \frac{\lambda^2}{6} \right)$.

The Eqs. (19) and (20) can be solved numerically. The explicit numerical solution is given by figures 1 and 2.

If the condition $R = 0$ is not satisfied, then one obtains the equation:

$$\begin{aligned} \frac{1}{\kappa_5^2} \left(1 - \frac{k_5^2 \lambda \alpha}{6} \right) R^{(4)} - 2 \left(\Lambda + \frac{k_5^2 \lambda^2}{6} \right) + \\ \alpha^2 \kappa_5^2 \left(\frac{1}{4} R_{\alpha\beta}^{(4)} R^{(4)\alpha\beta} - \frac{1}{12} R^{(4)2} \right) = 0. \end{aligned} \quad (22)$$

As an example, we discuss the situation when $a_1 = 0$. In this case the equation (22) accepts the form:

$$\begin{aligned} -\alpha^2 (e^{2a_2} - 1) k_5^4 + 2\alpha e^{2a_2} (e^{2a_2} - 1) k_5^2 \lambda r^2 + \\ 2e^{2a_2} r^2 (6 + e^{2a_2} [-6 + k_5^4 \lambda^2 r^2 + \\ 6k_5^2 \Lambda r^2]) + 2r (\alpha^2 (e^{2a_2} - 1) k_5^2 - \\ 12e^{2a_2} r^2 + 2\alpha e^{2a_2} k_5^2 \lambda r^2) \dot{a}_2 - \alpha^2 k_5^4 r^2 \dot{a}_2^2 = 0. \end{aligned}$$

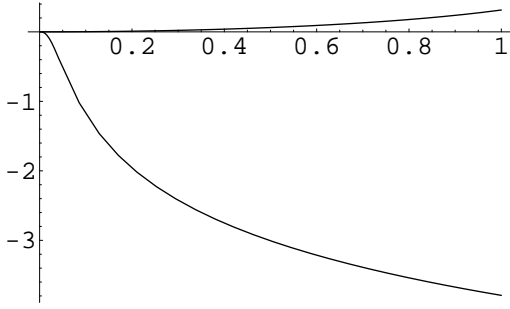


Figure 3: Evolution of the function $a_2(r)$. All constants are equal to one.

This equation can be solved numerically, as is drawn in figure 3.

Note if the bulk space is (Anti) deSitter (Weyl tensor is zero), one can obtain the following solution which describe 4d universe with constant curvature:

$$R_{\alpha\beta}^{(4)} = h q_{\alpha\beta}, \quad R^{(4)} = 4h. \quad (23)$$

$$h = \frac{6 - \alpha\kappa_5^4\lambda \pm \sqrt{6} \sqrt{6 - 2\alpha\kappa_5^4\lambda - \alpha_2\kappa_5^6\Lambda}}{\alpha^2\kappa_5^4}. \quad (24)$$

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